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Self-Consistent, Axisymmetric Two-Integral Models of Elliptical Galaxies with Embedded Nuclear Discs

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ABSTRACT

Ionized gas discs in the nuclei of ellipticals have proven to be excellent tools for the determination of the central mass density in these galaxies. The recent discovery with the Hubble Space Telescope of small *stellar* discs embedded in the nuclei of a number of ellipticals and S0s might be of similar importance. We construct two-integral axisymmetric models for such systems. The models consist of a spheroidal bulge with a central density cusp, and a disc described by a strongly flattened exponential spheroid. We use the Hunter & Qian (1993) method to calculate the even part of the phase-space distribution function (DF), and specify the odd part by means of a simple parameterization. We consider both local stability against axisymmetric perturbations, as well as global stability against bar forming modes, and find that our models are stable as long as the discs are not too flat and/or compact. The margin of stability is derived as a function of disc scalelength and central surface density. Its location agrees well with the observed values of these disc parameters. This suggests that discs build up their mass until they become marginally stable.

We investigate the photometric as well as the kinematic signatures of nuclear discs, including their velocity profiles (VPs), and study the influence of seeing convolution. In particular, we study to what extent these kinematic signatures can be used to determine the central density of the galaxy, and to test for the presence of massive black holes. We consider nuclear discs that are either dynamically coupled to or decoupled from the host elliptical, including counter-rotating discs. The latter are models for the counter-rotating cores observed in a number of galaxies, which are often found to exhibit discy isophotes in the central region. The counter-rotation is only detectable when the disc light contributes significantly to the central velocity profiles. We find that in this case the observed velocity dispersion will show a central decrease.

The rotation curve of a nuclear disc gives an excellent measure of the central mass-to-light ratio whenever the VPs clearly reveal the narrow, rapidly rotating component associated with the nuclear disc. Steep cusps and seeing convolution both result in central VPs that are dominated by the bulge light, and these VPs barely show the presence of the nuclear disc, impeding measurements of the central rotation velocities of the disc stars. However, if a massive BH is present, the disc component of the VP can be seen in the wing of the bulge part, and measurements of its mean rotation provide a clear signature of the presence of the BH. This signature is insensitive to the uncertainties in the velocity anisotropy, which often lead to ambiguity in the interpretation of a central rise in velocity dispersion as due to a central BH.

Key words: stellar dynamics – galaxies: kinematics and dynamics – galaxies: ellipticals – galaxies: structure – galaxies: nuclei – line: profiles

1 INTRODUCTION

Ever since the realization that elliptical galaxies are not smooth structureless systems of old stars supported by rotation, numerous studies have revealed the increasingly complex nature of these objects (de Zeeuw & Franx 1991). In addition to the bright, pressure supported triaxial systems with boxy isophotes, there are the discy ellipticals that are mainly supported by rotation. The central regions often exhibit complex structures. Counter-rotating cores, stellar cusps, and nuclear activity are all phenomena connected with the nuclei of elliptical galaxies (Kormendy & Richstone 1995). In addition, the nuclear regions might conceal massive black holes.

Numerous studies over the last decade have concentrated on the nature of the discy ellipticals (e.g., Capaccioli, Held, & Nieto 1987; Carter 1987; van den Bergh 1989; Nieto et al. 1991; Rix & White 1990, 1992; Scorza & Bender

1990). Scorza & Bender (1995) have shown that discy ellipticals follow the same trend as observed for spirals and S0s that discs with smaller scale lengths have higher central surface brightness (e.g., Kent 1985). This strongly suggests that discy ellipticals are two-component systems that form a continuing sequence in the Hubble diagram from S0s to galaxies with smaller disc-to-bulge ratios (see also Michard 1984; Capaccioli, Caon & Rampazzo 1990).

The Hubble Space Telescope (HST) has revealed very small (scale length $\sim$ 20pc), nuclear discs in a number of ellipticals (van den Bosch et al. 1994; Forbes 1994; Lauer et al. 1995). Several of these galaxies also harbour a much larger kpc-scale disc which may have an inner cut-off radius at $200 - 400$pc. Such multiple component discs have been observed in other galaxies as well (e.g., Seifert 1990; Scorza & Bender 1995). For example, Burkhead (1986) and Kormendy (1988) argue that the huge disc of the Sombrero galaxy (NGC 4594) consists of two separate components. Emsellem et al. (1994) even find an indication for a third disc component in the Sombrero, namely a red nuclear disc that remains unresolved from the ground.

In this paper we investigate the dynamical properties and observables of nuclear discs. We construct self-consistent axisymmetric dynamical models of a spheroidal bulge with an embedded nuclear disc. In particular we investigate whether the observable dynamical properties of nuclear discs can be used to discriminate whether or not these galaxies harbour a nuclear black hole (BH). Proving the presence of a BH in the nucleus of an elliptical galaxy is complicated by the fact that the three-dimensional velocity distribution of the stars in the central region can be strongly anisotropic. For example, the observed central increase of the velocity dispersion in M 87 can equally well be fitted by an isotropic model with a central BH as by a radially anisotropic model without any BH (Young et al. 1978; Sargent et al. 1978; Binney & Mamon 1982; van der Marel 1994). Discs, on the other hand, are cold components, strongly dominated by rotation. Furthermore, they are generally thin, so that the interpretation of the observed line-of-sight velocity distribution is less complicated than for the three-dimensional distribution of stars in the ellipsoidal component. Discs embedded in the nuclei of elliptical galaxies offer a promising means of detecting nuclear BHs. The recent discovery of a disc of ionized gas in the nucleus of M87 (Ford et al. 1994) has indeed greatly strengthened the case for a nuclear BH in this giant elliptical (Harms et al. 1994). Even more spectacular evidence for the presence of a nuclear BH is provided by the VLBA observations of emission of a disc of water masers with a Keplerian rotation curve at only 0.13 pc from the centre of NGC 4258 (Miyoshi et al. 1995). However, the interpretation of the dynamics of gas discs can be complicated, especially in AGNs where outflow, inflow, and turbulent phenomena are very likely to play a major role (see e.g., van den Bosch & van der Marel 1995; Jaffe et al. 1996). Since the motion of stars is purely gravitational, *stellar* discs do not suffer from these effects, and their kinematics might offer valuable tests of the presence of nuclear BHs. Recent absorption line spectroscopy with the Faint Object Spectrograph aboard the HST of the nucleus of NGC 3115, one of the galaxies with a bright nuclear, stellar disc, has indeed provided an excellent case for the presence of a $2 \times 10^9\,\mathrm{M}_\odot$ BH (Kormendy et al. 1996).

This paper is organized as follows. In Section 2 we describe our models. We then investigate the photometric observables of the nuclear discs in Section 3, with attention to their effect on the surface brightness profiles and isophotes. In Section 4 we construct both the even and the odd part of the DF. We study the stability of the resulting models in Section 5. In Section 6 we discuss the kinematic signatures of nuclear discs and the influence of seeing convolution. We investigate the influence of a nuclear black hole in Section 7, and summarize our results in Section 8.

2 MULTI-COMPONENT MODELS

We construct galaxy models consisting of a thick disc embedded in an spheroidal body: the 'bulge'. In addition, we allow a black hole to be present at the centre of the galaxy. The potential of the model can therefore be written as

$$\Psi(R,z) = \psi_{\rm bulge}(R,z) + \psi_{\rm disc}(R,z) + \psi_{\rm BH}(R,z). \tag{2.1}$$

We denote the *total* potential by $\Psi(R,z)$, and use $\psi(R,z)$ to indicate the potential of a separate component. We use the *relative* potential $\Psi = -\Phi$ instead of the potential Φ. Therefore stars at rest in the centre of the potential well have the *maximum* energy.

We define a disc to be *nuclear* if its horizontal scale length R_d is smaller than half the core radius R_b of the bulge. This is defined as the inner core- or break radius of the bulge component which $\sim 2'' - 3''$ at Virgo (Ferrarese et al. 1994), and should not be confused with the 'the Vaucouleurs' effective radius which is much larger. Although this paper only focuses on such nuclear discs, the method discussed here is also applicable to larger discs, and even to S0 galaxies, by simply increasing the ratio R_d/R_b.

Throughout this paper we take the mass-to-light ratio M/L to be equal to one, so that, e.g., the projected surface brightness is identical to the projected surface density.

2.1 The bulge component

We describe the spheroidal component by the so-called (α, β) models, which have a density distribution given by

$$\rho(R,z) = \rho_{0,b} \left(\frac{m}{R_b}\right)^{\alpha} \left(1 + \frac{m^2}{R_b^2}\right)^{\beta}, \tag{2.2}$$

where

$$m = \sqrt{R^2 + \frac{z^2}{q_b^2}}. \tag{2.3}$$

Here R_b is the 'break-radius' of the bulge, and q_b is the flattening of the bulge. For $\alpha = 0$ the central density is finite and equal to $\rho_{0,b}$. When $\alpha < 0$ the density profile has a central cusp. These models have proven to accurately fit the surface brightness distribution of elliptical galaxies (e.g., van der Marel et al. 1994, Qian et al. 1995 (hereafter QZMH), van den Bosch & van der Marel 1995). QZMH discuss these models in more detail.

At large radii the density (2.2) falls off proportional to $m^{\alpha+2\beta}$. We consider only models with $\alpha + 2\beta = -4$, so that the projected surface density at large radii falls off as R^{-3}. In this case the total mass is finite, and given by

$$M = 2\pi q_b \rho_{0,b} R_b^3 B\left(\tfrac{\alpha}{2} + \tfrac{3}{2}, \tfrac{1}{2}\right), \tag{2.4}$$

where B is the beta-function. These models are similar to the set of models discussed by Dehnen & Gerhard (1994), except that their density has $(1+m/R_b)^{2\beta}$ as second factor rather than $(1+m^2/R_b^2)^{\beta}$.

QZMH showed how the classical double quadrature expression (e.g., Chandrasekhar 1969) for the potential of any axisymmetric system in which the density is stratified on similar concentric spheroids, i.e., $\rho = \rho(m^2)$, can always be rewritten as a single quadrature. For our particular choice of $\alpha + 2\beta = -4$, however, the inner integration of the classical formula can be carried out explicitly, and therefore leads directly to a simple single quadrature, which is evaluated more efficiently. According to the classical formula, the potential that corresponds to (2.2) is given by

$$\psi(R,z) = \psi_{0,b} - \pi G q_b \int_0^\infty \frac{F(t)\mathrm{d}\tau}{(\tau+1)\sqrt{\tau+q_b^2}}, \tag{2.5}$$

where

$$F(t) = \int_0^{t^2} \rho(m'^2)\mathrm{d}m'^2. \tag{2.6}$$

Substitution of the density (2.2) in the above integral, and changing to the integration variable $x = \frac{(1+m^2/R_b^2)}{m^2/R_b^2}$, yields

$$F(t) = \frac{-\rho_{0,b} R_b^{2(\beta+2)}}{1+\beta} \frac{(1+t^2/R_b^2)^{\beta+1}}{t^{2(\beta+1)}}, \tag{2.7}$$

with

$$t = \sqrt{\frac{R^2}{\tau+1} + \frac{z^2}{\tau+q_b^2}}. \tag{2.8}$$

This result is valid when $\beta < -1$. Substitution of expression (2.7) in the integral (2.5) then gives the potential as a single quadrature. The central potential $\psi_{0,b}$ can be evaluated explicitly, and is given by

$$\psi_{0,b} = \frac{-GM}{(1+\beta)R_b B(\frac{\alpha}{2}+\frac{3}{2},\frac{1}{2})} \frac{\arcsin\sqrt{1-q_b^2}}{\sqrt{1-q_b^2}}. \tag{2.9}$$

2.2 The disc component

Recent work by, e.g., Scorza & Bender (1995), indicates that the small discs embedded in ellipticals form a smooth transition from S0s to galaxies with smaller disc-to-bulge ratios. We therefore assume that our nuclear discs have a surface brightness distribution that is similar to the discs of S0s and spirals. Most studies have considered models with infinitesimally thin discs. Although such discs provide a good approximation for some purposes, they are not sufficient for our needs, because they are maximally cold in the perpendicular (z-) direction. Real discs have non-zero z-motions that contribute to the line-of-sight velocities when observed at any inclination other than edge-on. Furthermore, infinitesimally thin models are unphysical in that the derivative $\partial\rho/\partial z$ is infinite at $z = 0$. We therefore consider *thick* discs. Observations have shown that the surface brightness of discs is well approximated by an exponential profile, both in the radial and the vertical direction (e.g., Freeman 1970; Aoki et al. 1991).

Few treatments of potential-density pairs of thick discs are available in the literature. Kuijken & Gilmore (1989) and subsequently Cuddeford (1993) presented a very simple method of expanding infinitesimally thin disc potentials to a thick disc with arbitrary vertical density distribution with constant scale height. Consider an infinitesimally thin disc, whose surface density is $\rho(R,z) = f(R)\delta(z)$, and with corresponding potential $\psi(R,z) = g(R,z)$. Here f and g are arbitrary functions. One can thicken this disc by replacing the Kronecker delta function $\delta(z)$ by any other function $h(z)$ that describes the vertical density distribution. At each z' parallel to the equatorial plane the density distribution can again be written with the delta function, i.e., $\rho(R,z') = f(R)\delta(z-z')$, and for this density distribution the corresponding potential is simply $\psi(R,z') = g(R,z-z')$. The potential of the thickened disc can be built up of an infinite number of such infinitesimally thin discs, each of which has a corresponding potential, properly weighted by the vertical density distribution. The potential

$$\psi(R,z) = \int_{-\infty}^{\infty} g(R,z-z')h(z')\mathrm{d}z' \tag{2.10}$$

follows straightforwardly.

Using the scheme outlined above, we could construct the potential density pair of the double exponential disc, i.e., $\rho(R,z) = \Sigma_0 \exp(-R/R_d)\exp(-|z|/z_0)$. The potential of the infinitesimally thin case is already a single quadrature, so that equation (2.10) becomes a double integration. For the special case of a double-exponential disc this can be reduced to a single quadrature (Kuijken & Gilmore 1989). This is an important advantage, since it speeds up the numerical calculations considerably. However, the double-exponential has the unphysical characteristic that the derivative $\partial\rho/\partial z$ is discontinuous at $z = 0$. Furthermore, the slow oscillatory behavior of the Bessel functions that occur in the integrand makes the numerical evaluation of the potential and its derivatives tedious.

We therefore decided to use an 'exponential spheroid', first introduced by Kent, Dame & Fazio (1991), to model the disc. This model, like the double-exponential disc, has an exponential surface brightness along both the radial and the vertical direction.

Consider a sphere whose projected surface brightness is

$$\Sigma(R) = \Sigma_0 \exp(-R/R_d). \tag{2.11}$$

The corresponding density distribution follows upon solving the Abel integral equation

$$\Sigma(R) = 2\int_R^\infty \frac{\rho r \mathrm{d}r}{\sqrt{r^2-R^2}}, \tag{2.12}$$

which gives

$$\rho(r) = \rho_{0,d} K_0(r/R_d), \tag{2.13}$$

where $\rho_{0,d} = \Sigma_0/\pi R_d$, and K_0 is a modified Bessel function.

We replace the sphere by a spheroid of flattening q_d, and introduce the spheroids $m = \sqrt{R^2 + z^2/q_d^2}$. This leads to axisymmetric models that project to a surface brightness that is exponential along any axis, i.e.,

$$\Sigma_i(x,y) = \frac{q_d}{q_d'}\Sigma_0 \exp(-R^*/R_d). \tag{2.14}$$

Here $R^* = \sqrt{x^2 + y^2/q_d'^2}$ and q_d' is the projected axis ratio, which is related to the intrinsic flattening and the inclination angle i through

$$q_d'^2 = \cos^2 i + q_d^2 \sin^2 i. \tag{2.15}$$

When $q_d \ll 1$ these models represent thick exponential discs.

The potential of the above discs can be evaluated by means of equation (2.5). Substitution of equation (2.13) in equation (2.6) gives

$$F(t) = -2\rho_{0,d} R_d t K_1(\frac{t}{R_d}), \tag{2.16}$$

so that the potential of the exponential spheroids is given by

$$\psi_d(R, z) = \frac{GM_D}{\pi R_d^2} \int\limits_0^\infty \frac{tK_1(\frac{t}{R_d})\mathrm{d}\tau}{(\tau+1)\sqrt{\tau + q_d^2}}, \tag{2.17}$$

where t is again given by equation (2.8), but with q_b replaced by q_d.

The potential can be written as a single quadrature with the modified Bessel function K_1 in the integrand. Since these functions decay exponentially and are positive definite, their numerical evaluation is much simpler than in the case of the double-exponential disc mentioned above. Furthermore, the derivative $\partial\rho/\partial z$ is continuous at $z = 0$.

The central density of the exponential spheroid is infinite, resulting from the fact that $\lim_{x\to 0} K_0(x) = -\ln(x)$. However, the total mass is finite; $M_D = 2\pi^2 q_d \rho_{0,d} R_d^3$. The central potential is given by

$$\psi_{0,d} = \frac{2GM_D}{\pi R_d} \frac{\arcsin(\sqrt{1 - \mathrm{q_d^2}})}{\sqrt{1 - \mathrm{q_d^2}}}. \tag{2.18}$$

2.3 The parameters

In order to constrain the large parameter space of our multi-component models we fix the values of a number of the parameters. Since we are mainly interested in the central parts of the models the only bulge-parameter that we vary is the cusp steepness α. The other parameters (R_b, q_b, and $\rho_{0,b}$) are kept fixed. The value of β is set by our requirement that $\alpha + 2\beta = -4$. We choose a flattening of 0.7 for the bulge.

For the disc we choose a fixed flattening of $q_d = 0.1$. This is consistent with the observed flattening of several nuclear discs (van den Bosch et al. 1994). Instead of parameterizing the disc by its mass $\mathrm{M_D}$ and scalelength R_d, we use the ratio R_d/R_b and the disc-to-bulge ratio Δ. We restrict ourselves to models with $R_d/R_b = 0.2$. This value is chosen to match approximately the values observed in a number of Virgo ellipticals (Scorza & van den Bosch, in preparation). Furthermore, taking $R_d/R_b = 0.2$ ensures that our choice to restrict ourselves to models with $\alpha+2\beta = -4$ does not influence the results. This is due to the fact that at radii where the slope of the density distribution of the bulge starts to depend on the particular choice of $\alpha + 2\beta$ (i.e., at $R \gtrsim R_b$), the contribution of the nuclear disk is already negligible.

We define Δ as the ratio of the total luminosity of the disc to the total luminosity of the bulge component. We assume that the mass-to-light ratios of both components are equal, and then have

$$\Delta \equiv \frac{M_{\mathrm{disc}}}{M_{\mathrm{bulge}}} = \frac{\pi}{B(\frac{\alpha}{2}+\frac{3}{2},\frac{1}{2})} \frac{q_d}{q_b} \frac{\rho_{0,d}}{\rho_{0,b}} \big(\frac{R_d}{R_b}\big)^3. \tag{2.19}$$

Table 1. Parameters of the models

Model	α	Δ	Λ	$M_{\mathrm{BH}}/M_{\mathrm{bulge}}$	$f_o(E, L_z)$	a
1.0	0.0	0.0	0.0	0.0	ST	+10
1.1	0.0	0.0146	1.0	0.0	ST	+10
1.2	0.0	0.0146	1.0	0.0	NR	+10
1.3	0.0	0.0146	1.0	0.0	NR	−10
1.4	0.0	0.0146	1.0	0.0	CR	+10
2.0	−0.5	0.0	0.0	0.0	ST	+10
2.1	−0.5	0.0207	1.0	0.0	ST	+10
3.0	−1.0	0.0	0.0	0.0	ST	+10
3.1	−1.0	0.0303	1.0	0.0	ST	+10
3.2	−1.0	0.0	0.0	0.00303	ST	+10
3.3	−1.0	0.0303	1.0	0.00303	ST	+10
3.4	−1.0	0.0	0.0	0.0303	ST	+10
3.5	−1.0	0.0303	1.0	0.0303	ST	+10
3.6	−1.0	0.0303	1.0	0.0	CR	+10
4.0	−1.5	0.0	0.0	0.0	ST	+10
4.1	−1.5	0.0444	1.0	0.0	ST	+10

For each model, the cusp-steepness α, the disc-to-bulge ratio Δ, the ratio of surface brightness for disc and bulge at $R = R_d$ (edge-on) Λ, and the ratio of BH mass over bulge mass are shown. In addition, the method and the value of the parameter a used to calculate the odd part of the DF are given. Here ST stands for the 'standard' models, NR for 'non-rotating bulge' models, and CR for 'counter-rotating' models (see Section 4.2).

Figure 1 shows a contour-plot of the potential and density in the meridional plane of a model with $\alpha = -1.0$, and $\Delta = 0.0303$. Although the presence of the nuclear disc is hardly visible from the potential contours, the flattened density contours in the centre clearly reveal its presence.

We also define the ratio of the surface brightness of disc and bulge at $R = R_d$, for an inclination angle of $i = 90^\circ$,

$$\Lambda \equiv \frac{\Sigma_{i=90}^d(R_d, 0)}{\Sigma_{i=90}^b(R_d, 0)}. \tag{2.20}$$

We choose the disc-to-bulge ratio Δ of our models in such a way that $\Lambda = 1.0$. In this way, the relative contribution of disc and bulge to the velocity profiles at $R = R_d$ is equal (for $i = 90^\circ$).

Table 1 summarizes the parameters of the entire set of models.

3 PHOTOMETRIC OBSERVABLES OF NUCLEAR DISCS

We first investigate the photometric characteristics of nuclear discs embedded in a spheroidal body. We project the models defined in Section 2 at different inclination angles i, and subsequently seeing-convolve them with a Gaussian Point Spread Function (PSF). The parameter $\chi \equiv \mathrm{FWHM}/R_d$ gives the FWHM of the PSF expressed in units of the horizontal disc scale length R_d. We scale the galaxy such that the FWHM corresponds to $1''$ in all cases, and take our pixels to be $0.2'' \times 0.2''$. After seeing-convolution we determine, as a function of radius, the ellipticity and the amplitude of the $\cos(4\theta)$-term in the Fourier expansion that describes the deviation of the isophote from a pure ellipse (e.g., Lauer 1985, Jedrzejewski 1987). If this amplitude is positive it indicates that the isophote is 'discy', whereas negative amplitudes imply 'boxy' isophotes.

Figure 2 presents the results. Since we define the FWHM of the PSF to equal $1''$, changing χ corresponds

to changes in the scale of the galaxy, i.e., its distance. We plot all parameters out to $10''$. The presence of the nuclear disc is evident from the central increase in ellipticity, and the disciness of the isophotes in the central region. When the bulge harbours a steep cusp the main photometric signature of the nuclear disc is the disciness of the isophotes, as can be seen in the panels on the left. The panels in the middle of Figure 2 show that the nuclear discs become photometrically undetectable if the FWHM of the PSF exceeds $2-3$ times the disc scale length. Furthermore, as was already shown by Rix & White (1990), the discs become photometrically undetectable for $i \lesssim 60^\circ$ (panels on the right).

When trying to decompose the surface brightness distribution in a disc and a bulge component from the photometry alone, a number of indicators can be used. The radius at which the $\cos 4\theta$ profile reaches its maximum is strongly correlated with the scale length R_d of the disc (Scorza & Bender 1995). The left panels in Figure 2 illustrate that the central ellipticity profile contains information on the cusp steepness α. The amplitude of the central changes in the ellipticity and $\cos 4\theta$ profiles are related to the disc-to-bulge ratio (or equivalently, the central surface brightness of the nuclear disc). All these indicators also depend on the inclination angle i. Scorza & Bender (1995) showed that the $\cos 6\theta$ parameter can be used to constrain i, under the assumption that the disc is infinitesimally thin. However, there is a degeneracy between disc thickness and inclination angle, because our discs are stratified on similar concentric spheroids. This will lead to some non-uniqueness for the disc and bulge parameters, but the solution space will be small for sufficiently inclined galaxies.

4 CONSTRUCTION OF THE $F(E, L_Z)$ MODELS

Any collisionless system can be fully specified by its distribution function, which, as stated by the Strong Jeans Theorem, depends on the phase-space coordinates only through its isolating integrals of motion. Most stellar orbits in an axisymmetric potential admit three such isolating integrals, namely the energy E, the angular momentum along the rotation axis L_z, and some third integral I_3. Except for Stäckel potentials (e.g., de Zeeuw 1985), the non-classical third integral is generally not known analytically, and may not exist for all orbits. This is the reason that most axisymmetric models constructed to date have distribution functions that depend only on the two classical integrals of motion; $f = f(E, L_z)$. Although such models often predict too much motion on the major axis due to the fact that the velocity dispersions in the R- and z-directions are everywhere equal, $\sigma_R = \sigma_z$, they provide reasonable approximations to the dynamics of some galaxies (van der Marel 1991; QZMH; Dehnen 1995). here we also limit ourselves to two-integral models. We discuss the limitations of this approach in Section 8.

4.1 The even part of the distribution function

The density that corresponds to the combined potential $\Psi(R, z)$ of bulge, nuclear disc, and BH, depends only on the part of the DF that is even in L_z:

$$\rho = \frac{4\pi}{R} \int_0^{\Psi} \mathrm{d}E \int_0^{R\sqrt{2(\Psi-E)}} f_e(E, L_z)\mathrm{d}L_z. \tag{4.1}$$

For $L_z = 0$ the inverse of this equation is given analytically by Eddington's formula

$$f_e(E,0) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^E \frac{\mathrm{d}^2\rho}{\mathrm{d}^2\Psi}\frac{\mathrm{d}\Psi}{\sqrt{E-\Psi}} + \frac{1}{\sqrt{E}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\right)_{\Psi=0}\right], \tag{4.2}$$

but the evaluation of $f_e(E, L_z)$ for other values of L_z from a given $\rho(R, z)$ is not easy. Lynden-Bell (1962), Hunter (1977), and Dejonghe (1986) used different transformation methods to evaluate $f_e(E, L_z)$ for a number of models. However, since all these methods require the analytical knowledge of $\rho(\Psi, R)$, they are not widely applicable.

Recently more general schemes have become available for the evaluation of $f_e(E, L_Z)$. Hunter & Qian (1993, hereafter HQ) found a contour integral expression, which is the generalization of Eddington's formula. Other methods for generating $f_e(E, L_z)$ were developed by Dehnen (1995), Magorrian (1995), and Kuijken (1995). None of these methods requires that the density can explicitly be written as a function of Ψ, so they allow the evaluation of $f_e(E, L_z)$ for any $\rho(R, z)$. QZMH and also Dehnen (1995) successfully used these methods to construct a two-integral model of M32 with a central black hole that provides an accurate fit to the observed velocity profiles.

We use the HQ-method. The HQ solution for the even part $f_e(E, L_z)$ of the DF is

$$f_e(E, L_z) =$$

$$\frac{1}{\sqrt{8}\pi^2 i} \int_{\Psi_\infty}^{[\Psi_{env}(E)+]} \tilde{\rho}_{11}\left[\xi, \frac{L_z^2}{2(\xi - E)}\right] \frac{\mathrm{d}\xi}{\sqrt{\xi - E}}. \tag{4.3}$$

Here the integral is along a complex contour in the plane of the complex potential ξ. The two subscripts of $\tilde{\rho}_{11}$ denote the *second* partial derivative with respect to the *first* argument (ξ). The tilde denotes the complex continuation of the function ρ_{11}, which is given by

$$\rho_{11}(\xi, R^2) = \frac{\rho_{22}(R^2, z^2)}{[\Psi_2(R^2, z^2)]^2} - \frac{\rho_2(R^2, z^2)\Psi_{22}((R^2, z^2)}{[\Psi_2(R^2, z^2)]^3}. \tag{4.4}$$

Each subscript 2 denotes the partial derivative with respect to z^2. The HQ paper gives further details on the method. The complete procedure of how to calculate the complex contour integral is discussed in detail in QZHM.

We split $f_e(E, L_z)$ in two parts; a bulge part $f_e^b(E, L_z)$, and a disc part $f_e^d(E, L_z)$. Each of these two terms of $f_e(E, L_z)$ is calculated by solving equation (4.3), replacing $\rho \equiv \rho_{\rm bulge} + \rho_{\rm disc}$ with $\rho_{\rm bulge}$ and $\rho_{\rm disc}$ respectively. The potential remains the *total* potential $\Psi \equiv \psi_{\rm bulge} + \psi_{\rm disc}$.

We evaluate $f_e(E, L_z) = f_e^b(E, L_z) + f_e^d(E, L_z)$ on a 40×40 grid in the region of the (E, L_z^2)-plane that corresponds to bound orbits in the potential Ψ. This region is bounded by $L_z \geq 0$, $E \leq \Psi_\infty$, and by the curve $L_z \leq L_{z,\max}(E)$. Here $L_{z,\max}(E)$ is the maximum allowed value of L_z for energy E, and corresponds to the angular momentum of a star with energy E on a circular orbit. For the 40 grid points that have zero angular momentum we compare the result of the complex contour integration with the value given by Eddington's formula (4.2), to check that the integral on the chosen contour converges, and whether any adjustments of the contour are necessary. We check for self-consistency by

calculating $\rho(R,z)$ from the DF, using equation (4.1). Bicubic spline interpolation is used to interpolate between grid points. The models are self-consistent to a high degree of accuracy ($\sim 10^{-3}$).

Figure 3 shows $\log_{10}[f_e(E,L_z)]$ as function of E/Ψ_0 and $\eta^2 \equiv [L_z/L_{z,\max}(E)]^2$ for four different models: model 1.0 (upper left), model 1.1 (lower left), model 3.0 (upper right), and model 3.1 (lower right). See Table 1 for the parameters of each of these models. The 40×40 grid is also shown. In order to properly sample the DF, the grid-density increases with energy and angular momentum. The presence of a central cusp in the bulge density is evident from a strong rise in the centre (i.e., at large E/Ψ_0). The presence of the nuclear disc also induces such a strong rise. Furthermore, close to the centre the DF is peaked towards high angular momentum, indicative of the rotational support of the disc.

4.2 The odd part of the distribution function

Whereas the even part of the DF generates the density of the system through equation (4.1), the odd part of the DF specifies the mean streaming motion:

$$\rho\langle v_\phi\rangle = \frac{4\pi}{R^2}\int\limits_0^{\Psi} \mathrm{d}E \int\limits_0^{R\sqrt{2(\Psi-E)}} f_o(E,L_z)L_z \mathrm{d}L_z. \qquad (4.5)$$

Therefore, only if in addition to $\rho(R,z)$, the *entire* mean streaming of the stars is known *a priori*, can one obtain $f_o(E,L_z)$ by inversion of equation (4.5). Since we are primarily interested in studying the observable properties of nuclear discs for different amounts of rotational support, we exploit the freedom to specify $f_o(E,L_z)$ under the constraint that $f(E,L_z) \equiv f_e(E,L_z) + f_o(E,L_z)$ be positive.

We consider three different approaches for specifying $f_e(E,L_z)$. In the first approach we define a function $h_a(\eta)$, where $\eta \equiv L_z/L_{z,\max}(E)$, and simply take

$$f_o(E,L_z) = h_a(\eta)\, f_e(E,L_z), \qquad (4.6)$$

By doing so we consider the entire system as a *one-component* model, i.e., although the potential is provided by the disc and the bulge, the two components are dynamically coupled to each other since we do not treat the streaming motions in them separately. In order to limit the number of free parameters we consider a $f_o(E,L_z)$ that is fully specified by only one parameter. Dejonghe (1986) showed that there is such a functional form which has the advantage that it maximizes the entropy of a two-integral axisymmetric system. We follow van der Marel et al. (1994), and use a modified version of this parameterization, which is given by

$$h_a(\eta) \equiv \begin{cases} \tanh(a\eta/2)\,/\,\tanh(a/2) & (a>0) \\ \eta & (a=0) \\ (2/a)\mathrm{arctanh}\big[\eta\tanh(a/2)\big] & (a<0) \end{cases} . \qquad (4.7)$$

The other advantage of this parameterization is that these models will automatically be physical (i.e., have positive DF) as long as $f_e(E,L_z) > 0$. The free parameter a determines the amount of rotation, whereby the model of maximum rotation has $a = \infty$. Van der Marel et al. (1994) and also QZMH used an extended version of this parameterization in which they included another parameter that determines the fraction of stars on clockwise, circular orbits in the equatorial plane. Although this allows the study of more general models, it invokes yet another free parameter. Therefore, we restrict ourselves to models in which this fraction is unity, i.e., in which all the stars on *circular* orbits in the equatorial plane have the same rotation direction.

The second approach allows us to construct models with a non-rotating bulge and a rotating disc. We calculate $f_e^b(E,L_z)$ and $f_e^d(E,L_z)$ separately, and then define the odd part of the distribution function of disc and bulge separate from each other. This gives models that consist of two (dynamically) decoupled components. We consider cases in which $f_o^b(E,L_z) = 0$ (no rotation of the bulge component), and

$$f_o^d(E,L_z) = h_a(\eta)\, f_e^d(E,L_z). \qquad (4.8)$$

Here $h_a(\eta)$ is defined by equation (4.7).

Finally, in our third approach for specifying $f_o(E,L_z)$ we construct models with a counter-rotating disc by simply taking

$$f_o^b(E,L_z) = +h_a(\eta)\, f_e^b(E,L_z), \qquad (4.9)$$

and

$$f_o^d(E,L_z) = -h_a(\eta)\, f_e^d(E,L_z), \qquad (4.10)$$

Throughout the remainder of this paper we will refer to models constructed with the first method (i.e., with dynamically coupled components) as 'standard models'. The models for which the odd part of the distribution function of the bulge is zero as 'models with non-rotating bulge', and the models whose odd part results in counter-rotating discs as 'counter-rotating models'.

4.3 The region of physical models

QZMH showed that two-integral oblate (α,β)-models with a nuclear BH are unphysical (i.e., have a region in phase-space where the DF is negative) when $\alpha > -0.5$. Therefore, it is to be expected that models with very compact, massive nuclear discs embedded in an (α,β)-bulge, with $\alpha > -0.5$ will also be unphysical. Since a nuclear disc is a cold component (see also Section 5.2) located at the centre of the galaxy, most stars in the centre will be on nearly circular orbits. In the derived DF this translates into small weights for the more radial orbits close to the centre. This can clearly be seen from the DF of model 1.1 in Figure 3. The DF is peaked towards large energies (i.e., the centre of the potential well) and high angular momentum. A local minimum close to the centre at $L_z = 0$ is also visible.

The value of the DF at this local minimum depends on the disc-to-bulge ratio of the system. The models will become unphysical if $\Delta = M_{\rm disc}/M_{\rm bulge}$ is too large, since this will bring about a region in phase-space were $f_e(E,L_z)$ is negative. If Δ is increased further the nuclear disc becomes completely dominant over the bulge. In those cases the DF is positive again over the entire phase space. So as function of R_d there is an interval of disc-to-bulge ratios for which the DF has a local minimum with negative values. Since the angular momentum at this local minimum is zero, one can test whether a certain $(R_d/R_b,\Delta)$-model is physical by examining whether $f_e(E,L_z=0)$ is positive for all $E < \Psi_0$. In order to determine this interval we proceed as follows. For each model $(R_d/R_b,\Delta)$ we numerically

evaluate $E_{\rm crit}$ for which $f_e(E, L_z = 0)$ is (locally) minimal, using Brent's method for the minimization. Subsequently, we check whether $f_e(E_{\rm crit}, L_z = 0) < 0$. We repeat this procedure in a bisective way until the borders of the interval are known to a certain accuracy. The values of $f_e(E, L_z = 0)$ are evaluated using Eddington's formula (4.2).

The results for the case with $\alpha = 0$ and $\alpha = -0.25$ are shown in Figure 4, where we plot $\log_{10}(\Delta)$ as function of $\log_{10}(R_d/R_b)$. The hatched areas indicate the regions where the models are unphysical. Only very small, massive nuclear discs give rise to unphysical two-integral DFs. As expected, the region of unphysical models becomes smaller when the cusp steepness increases (i.e., when the value of α decreases).

5 THE STABILITY OF THE NUCLEAR DISCS

Our $f(E, L_z)$-models are of little practical value if they are unstable, and hence we study their stability in this section.

5.1 Local stability

The best known stability criterion for discs is the Toomre (1964) criterion. This criterion ensures that an infinitesimally thin disc is *locally* stable against axisymmetric perturbations as long as

$$Q \equiv \frac{\sigma_R \kappa}{\gamma G \Sigma} > 1. \tag{5.1}$$

Here σ_R is the radial velocity dispersion, Σ is the surface density of the disc, γ is a constant that equals 3.36, G is the gravitational constant, and κ is the epicyclic frequency.

In an axisymmetric system with distribution function $f_e(E, L_z)$ we have $\sigma_R = \sigma_z = \sigma$ everywhere. From the Jeans equations it follows that (e.g., Hunter 1977)

$$\sigma^2(R, z) = -\frac{1}{\rho(R, z)} \int_z^\infty \rho(R, z') \frac{\partial \Psi}{\partial z}(R, z') \mathrm{d}z'. \tag{5.2}$$

The surface density $\Sigma(R)$ of our disc is given by

$$\Sigma(R) = \int_{-\infty}^{\infty} \rho_{\rm disc}(R, z) \mathrm{d}z. \tag{5.3}$$

A problem that arises is that we are considering a thick disc. Only a limited amount of work has been done on the stability of thick discs, but the little work that has been done (e.g., Shu 1968; Vandervoort 1970) indicates that disc thickness has a stabilizing effect. This means that the factor $\gamma = 3.36$ derived by Toomre for the infinitesimally thin case decreases for thick discs. However, it is not known what the value of γ is for an arbitrary vertical density profile. Therefore, instead of presenting $Q(R)$, we show the radial profile of $\gamma\, Q(R)$.

The results are shown in Figure 5, where we give γQ as function of radius for four models that differ only in cusp steepness. The hatched region indicates the regime where infinitesimally thin discs are unstable against axisymmetric perturbations (i.e., $\gamma Q \le 3.36$). In the case of a thick disc, the upper limit of this regime will move slightly downwards. The presence of a cusp stabilizes the disc in the inner region. Only the nuclear disc in the model without cusp ($\alpha = 0.0$) is mildly unstable in its inner region. Figure 6 shows the region in the (α, Δ)-plane of models that are stable over the entire radial range (hatched region), where we have assumed that stability is achieved for $\gamma Q \ge 3.0$. Although this value is uncertain, as we have seen, a 10% change will cause only a very small shift in the location of the boundary curve. It is again evident that a cusp has a strong stabilizing effect on a nuclear disc.

Although the surface density of the disc falls off strongly, the velocity dispersion decreases slowly, due to the fact that there is a large amount of matter (from the bulge-component) beyond several scale lengths of the disc. Therefore, the outer parts of the disc are stable in all cases.

Figure 7 shows γQ as function of radius for the cusp-free model with $\Lambda = 1.0$ (model 0.1), for three different flattenings of the nuclear disc. Thickening the disc, i.e., increasing q_d, has a stabilizing effect. This can also be understood intuitively: the thickness of a disc is closely related to the amount of random motion of the stars in the z-direction. Since our models have DFs that depend on only two integrals of motion, we have $\sigma_R = \sigma_z$. Therefore, thickening the disc will increase the radial velocity dispersion in the disc, and therefore the parameter Q. The dependence of κ on q_d is weak.

5.2 Global stability

Besides *local* instabilities, discs can also suffer from *global* instabilities. The best known is the bar-instability, which has proven to play an important role. Ostriker & Peebles (1973) calculated that a disc is globally stable to bar-like modes when the total kinetic energy of rotation T is less than $\sim$ 14% of the gravitational energy $|W|$. However, this criterion is merely a good rule of thumb, rather than a proven physical theorem. Athanassoula & Sellwood (1986) showed that, even more effectively than a halo component, random motions in the disc can stabilize it against bar forming modes. They showed that as long as the mass-weighted value of Q averaged over the central region exceeds a value of $\sim 2 - 2.5$, the entire disc is stable. This is the case for all of the models listed in Table 1.

Toomre (1981) argued that the origin of the bar forming instability is feedback to the swing amplifier via waves that pass through the centre. The occurrence of swing amplification depends on the value of

$$X = \frac{\kappa^2 R}{2 m \pi \Sigma}, \tag{5.4}$$

where $m = 2$ in the case of bi-symmetric bars. Toomre showed that as long as $Q \lesssim 3$ and $X \lesssim 3$ the gain of the swing-amplifier is sufficient to form bars. We find that X increases strongly with radius, but $X \lesssim 3$ for small R. Although this implies that swing amplification can occur in the very centre of the nuclear discs, one can ensure stability against bar formation by cutting off the path to the centre by means of an inner Lindblad resonance. The models with a cusp ($\alpha < 0$) have $\Omega - \frac{1}{2}\kappa \to +\infty$ for $R \to 0$, and will therefore have an ILR for any pattern speed of the bar. An ILR is absent only when $\alpha = 0$ and high pattern speeds. In that case the swing amplified feed-back loop is open in the centre ($X \lesssim 3$), but as there is a large amount of random motion in the centre (i.e., $Q > 3$) the growth rate for a bar

mode will be insignificant. Therefore, we expect that the nuclear discs in all the models listed in Table 1 are stable against bar formation.

The above results are valid for a bulge with axis ratio of 0.7. N-body experiments carried out by Dehnen (priv. comm.) indicate that non-rotating and rotating (α,β)-models with axis ratios of 0.5 or smaller are unstable to bar formation.

5.3 Continuation to larger discs

We have restricted ourselves to nuclear discs with $R_d/R_b = 0.2$, but the method outlined above is also applicable to larger discs. We have investigated the stability of our disc-models for a large range in R_d/R_b. The general tendency is for larger discs to be more stable. We have calculated the maximum disc-to-bulge ratio for which, at given cusp-steepness α and disc scale length R_d, $\gamma Q(R) \geq 3.0$. From equation (2.19) it follows that $\Sigma_0 \propto \Delta/R_d^2$. Therefore, for given disc-to-bulge ratio and disc-scale length we can, using an arbitrary scaling, calculate the central surface brightness μ_0 of the disc (in magnitudes per arcsec^2).

Figure 8 shows the curve of μ_0 as a function of $\log_{10}[R_d/R_b]$ for discs with maximum disc-to-bulge ratio so that the entire disc has $\gamma Q \geq 3$. These curves depend on the cusp steepness α, but only for small discs. This Figure is remarkably similar to the relation between μ_0 and R_d of observed discs (i.e., compare to Figure 17 of Scorza & Bender 1995). This might indicate that discs build up their mass until they become marginally stable.

6 THE KINEMATICS OF NUCLEAR DISCS

Our models are fully specified by the four parameters α, Δ, a, and $M_{\rm BH}$. In addition, we must choose the method used to calculate the odd part of the DF. In this section we restrict ourselves to cases with $M_{\rm BH} = 0$. The influence of a nuclear BH in the centre of these models is the topic of Section 7.

6.1 Calculation of the velocity profiles

Once we have determined the complete phase-space density of our models, we can calculate every observable property. In particular, the entire VP can be derived by integrating the DF along the line of sight:

$$\mathrm{VP}(v_{z'};x',y') = \frac{1}{\Sigma} \int \int_{E>\Psi_\infty} \int f(E,L_z) \mathrm{d}v_{x'} \mathrm{d}v_{y'} \mathrm{d}z'. \qquad (6.1)$$

Here we adopt the Cartesian coordinate system of an observer (x',y',z'), where the z'-axis lies along the line of sight, and x' is along the apparent major axis of the galaxy projected on the sky.

Upon defining a polar coordinate system $(v_\perp,\varphi)$ in the $(v_{x'},v_{y'})$ plane, where $v_{x'} = v_\perp \cos\varphi$, and $v_{y'} = v_\perp \sin\varphi$, equation (6.1) can be written as

$$\mathrm{VP}(v_{z'};x',y') = \frac{1}{\Sigma} \int_{z_1}^{z_2} \mathrm{d}z' \int_0^{2\Psi - v_{z'}^2} \mathrm{d}v_\perp^2 \int_0^\pi f(E,L_z)\, \mathrm{d}\varphi. \qquad (6.2)$$

The energy is given by

$$E = \Psi(x',y',z') - \frac{1}{2}(v_{z'}^2 + v_\perp^2), \qquad (6.3)$$

and the angular momentum follows from

$$L_z = -v_{z'} x' \sin i$$

$$+ v_\perp \cos\varphi \sqrt{(-y'\cos i + z'\sin i)^2 + x'^2 \cos^2 i}. \qquad (6.4)$$

For any given $v_{z'}$ the boundaries z_1 and z_2 are determined by solving $\Psi(x',y',z') - \frac{1}{2}v_{z'}^2 = 0$.

For each model we calculate the VPs at a number of (x',y')-points. In all cases discussed here we take an inclination angle of 90°, i.e., we assume edge-on observation. Figures 9a and 9b compare the VPs of model 1 ($\alpha = 0.0$) at seven positions along the major axis ($x' = R, y' = 0$), using four different odd parts of the distribution function: the first (left panels in Figure 9a) has $f_o(E,L_z)$ specified by the 'standard' method (Section 4.2). The panels on the right in the same figure correspond to a 'non-rotating bulge' model with $a = +10$, whereas taking $a = -10$ results in the VPs shown in the left panels of Figure 9b. Finally, in the right panels of Figure 9b, the VPs are shown that correspond to the case of a counter-rotating disc, i.e., where we have reversed the sign of the odd part of the DF of the disc with respect to that of the bulge.

Even though the *dynamics* of bulge and disc are coupled in the 'standard' models, the VPs nevertheless show two very distinct components. The solid triangles in Figure 9 indicate the circular velocity of the models. As can be seen, the disc rotates with almost this circular velocity. This is true for both the 'standard' models, the 'non-rotating bulge' models, and the 'counter-rotating' models.

As discussed in Section 4.2, we have constrained the fraction of stars on clockwise circular orbits in the equatorial plane to be unity. However, not all stars in the disc are on such circular orbits. Therefore, upon taking $a \ll 0$ in the odd part of the DF, a considerable fraction of stars in the disc will be on anti-clockwise orbits that are not (perfectly) circular. This results in double-peaked VPs, as can be seen in the left panels of Figure 9b.

6.2 Velocity profile analysis

In order to allow for an easy comparison between different models, and to be able to extract physically meaningful parameters, we need to find a way of parameterizing the VPs. It has become customary to expand the VPs in a Gauss–Hermite series (van der Marel & Franx 1993; Gerhard 1993). Although this approach has proven useful when parameterizing VPs that deviate slightly from a Gaussian, it is inappropriate for describing the VPs presented above, since they strongly deviate from Gaussians. After seeing convolution with a FWHM that exceeds a few disc scale-lengths the Gauss–Hermite parameterization *can* be used, since the convolution results in sufficiently smooth VPs. However, for smaller seeing FWHM the convolved VPs still deviate strongly from a Gaussian.

We therefore opted to use the real moments to quantify the VPs. The $n^{\rm th}$ moment of the normalized velocity profile

$\mathrm{VP}_0(v)$ is defined as

$$\mu_n \equiv \int_{-\infty}^{\infty} \mathrm{VP}_0(v)\, v^n \, \mathrm{d}v. \tag{6.5}$$

The mean streaming rotation velocity is simply the first moment $V_{\rm rot} = \mu_1$, the velocity dispersion $\sigma = \sqrt{\mu_2 - \mu_1^2}$, the skewness $\mathcal{S} = (\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3)/\sigma^3$, and the kurtosis $\mathcal{K} = (\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4)/\sigma^4$.

Figure 10 shows these parameters for models 1.1 – 1.4 whose VPs are given in Figures 9a and 9b. The presence of the nuclear discs brings about VPs that deviate strongly from a Gaussian. This is evident from the large values of both $\mathcal{S}$ and $\mathcal{K}$, which are zero and three respectively for a Gaussian. The disc dominates the kinematics in the centre, and causes a strong drop of the velocity dispersion inwards of $\sim 3R_d$. The rotation curve of model 1.4 clearly reveals a counter-rotating core. This is also evident from the change of sign of the skewness $\mathcal{S}$ around $\sim 2.5R_d$.

The real moments of the VPs describe the dynamics of the entire system. If one is interested in the dynamics of disc and bulge separately, another parameterization of the VPs is required. Given the clear two-component character of the VPs, it is convenient to use the Levenberg-Marquardt method to fit a double Gaussian to the derived VPs. Under the assumption that the two separate components of the VPs correspond to the disc and bulge components of the model, and assuming that both are Gaussian, the double-Gaussian fit measures the kinematics of disc and bulge separately (Rix & White 1992). There is no *a priori* reason why the VPs of both components will be Gaussian. However, increasing the number of parameters of the fitting function will result in a large degeneracy of the parameters. Furthermore, Scorza & Bender (1995) have shown that including deviations from a Gaussian only mildly influences the derived kinematics of bulge and disc.

6.3 Observations of nuclear discs

We now discuss the influence of seeing convolution on the observable properties of nuclear discs. We mimic long-slit observations where we take the slit to be aligned along the major axis, the slit-width to be equal to the FWHM of the seeing, and the pixels to be one-third of the seeing FWHM. We use the parameter χ to define the PSF FWHM, in units of the scale-length R_d of the disc, and subsequently scale the models so that the FWHM equals $1''$. Therefore varying χ is similar to varying the distance of the galaxy from the observer. We start by constructing a cube that contains the velocity profiles $\mathrm{VP}(v_{z'}; x', y')$ calculated on a two-dimensional logarithmic grid on the sky (x', y'). We then convolve the velocity profiles for each $v_{z'}$ with a Gaussian PSF and take the effects of slit-width and pixel size into account. We use bi-cubic spline interpolations to interpolate between (x', y') grid points at a given velocity $v_{z'}$. We use the approach taken by QZMH, who have shown that in the case of a Gaussian PSF and rectangular pixels, the entire seeing convolution and pixel averaging can be written as a double quadrature. After the convolution we parameterize the VPs. Since the convolved VPs deviate strongly from a Gaussian for small values of χ, we use the moments defined in the previous section to describe the VPs.

Figure 11 shows the results for four different values of the cusp steepness α. They include the case without nuclear disc (solid lines) as well as the case with nuclear disc with disc-to-bulge ratio such that $\Lambda = 1.0$ (dashed lines). The odd part is determined in the 'standard' way with $a = 10$. In all cases we take an inclination angle of $i = 90^{\rm o}$, and convolve the model with a PSF with $\chi = 1$. The main effect of changing α is that the rotation curve of the entire galaxy becomes steeper, due to larger central mass. The kinematic signature of a nuclear disc is therefore most pronounced in models with steep cusps. The seeing convolution results in velocity dispersions that are almost equal for the cases with and without nuclear disc. The strong central decrease of dispersion (as can be seen in Figure 10) is no longer observable for a seeing with FWHM equal to the scale-length of the nuclear disc. The kinematic signatures of the nuclear disc are confined to higher rotation velocities and stronger deviations of the VPs from a Gaussian.

Figure 12 shows in more detail the effect of the seeing convolution. Here we plot the moments of the convolved VPs for the model with $\alpha = -1.0$, again both with and without nuclear disc (models 3.0 and 3.1 respectively), for four different values of χ. Upon increasing χ, the VPs of the models with and without nuclear discs become more and more similar. For $\chi \gtrsim 3$ the only kinematic signatures of the nuclear disc that remain are a slightly larger maximum rotation velocity and central velocity dispersion. However, the differences are so small that it is unlikely that the presence of a nuclear disc can be inferred from kinematic observations with a seeing FWHM exceeding ~ 3 times the disc scale-length. This is also the maximum χ beyond which the disc becomes *photometrically* undetectable. Figure 13 summarizes these results for the velocity dispersion. We give the ratio of central velocity dispersion for the case with nuclear disc ($\Lambda = 1$) to the case without nuclear disc ($\Lambda = 0$) as function of χ for four different values of cusp steepness. Increasing α results in an increase of the central mass, and therefore in an increase of the central rotation velocity. Increasing χ results in an increase of the ratio $\sigma_0(\Lambda = 1)/\sigma_0(\Lambda = 0)$ due to seeing convolution of the rotation curve.

A number of authors have suggested that the presence of a nuclear disc can mimic the presence of a BH, in that seeing convolution of the central rotation curve might result in an increase of the central velocity dispersion. If the PSF FWHM is so large that the disc is photometrically undetectable, the large central velocity dispersion could erroneously be interpreted as due to a BH. However, we are unable to achieve a ratio of $\sigma_0(\Lambda = 1)/\sigma_0(\Lambda = 0)$ larger than 1.1. This is due to the fact that the PSF is circular compared to the highly flattened disc structure. Upon reducing the slit-width we only find a very mild increase of $\sigma_0(\Lambda = 1)/\sigma_0(\Lambda = 0)$. The only way of achieving a large enough ratio so that the nuclear disc might mimic a BH is by strongly increasing α and the disc-to-bulge ratio Δ. However, stability arguments prohibit too large values of Δ. Furthermore, increasing Δ will make the nuclear disc detectable in the photometry.

The entire analysis above is based on edge-on projections of the models. Changing the inclination angle has a number of effects. First of all, since the disc is highly flattened, the observed rotation of the disc roughly scales with $\sin i$. Furthermore, decreasing the inclination angle

decreases the surface brightness of the disc relatively more than that of the bulge component. As a consequence, decreasing i results in VPs that less and less clearly show a rapidly rotating component. We projected model 3.1 at a number of different inclination angles and found the kinematic signatures of the nuclear disc to disappear for $i \lesssim 60^\circ$.

6.4 Counter-rotating discs

In Section 4.2 we discussed how to define the odd part of the DF in order to achieve a model in which the nuclear disc counter-rotates with respect to the bulge. Several galaxies have been observed to harbour counter-rotating cores. Recent imaging with the Hubble Space Telescope has shown that in a number of cases there are indications for the presence of a disc component in the centre (Forbes, Franx, & Illingworth 1995, Carollo et al. 1996). Therefore, counter-rotating discs seem a natural explanation for the observed kinematics. Here we investigate the kinematic signatures of such counter-rotating nuclear discs.

We find that for $\chi \lesssim 2$ the VPs are double-peaked, clearly revealing the counter-rotation. In these cases one observes also a strong central decrease of the velocity dispersion. This is due to the fact that the dynamically cold disc component dominates the central VPs, and to the fact that, at larger radii, the second moment of the VP becomes large due to the increasing separation of the disc- and bulge-components of the VPs. It is remarkable that in most galaxies with a counter-rotating core, the central velocity dispersion strongly *increases* towards the centre, e.g., IC 1459 (van der Marel & Franx 1993), NGC 3608 (Jedrzejewski & Schechter 1989). More extensive modeling of these systems, possibly with a nuclear BH, is clearly required.

For cases with $\chi \gtrsim 2$, the only kinematic signature of the counter-rotating disc are strongly skewed VPs. However, the interpretation of these as being due to a counter-rotating disc is ambiguous.

6.5 Deriving the central mass density

The mean streaming of the disc stars follows from the odd part of the DF by means of equation (4.5), and substituting ρ_d and f_o^d for ρ and f_o respectively. This mean streaming can be compared to the circular velocity, which is given by

$$v_{\rm circ}^2(R) = -R\frac{d\Psi}{dR} = 4\pi G q \int\limits_0^R \frac{\rho(m^2) m^2 dm}{\sqrt{R^2 - m^2(1-q^2)}}. \qquad (6.6)$$

As shown in Section 6.1, the disc rotates with nearly the circular velocity ($\bar{v}_\phi$ of the disc stars $\gtrsim 0.95 v_{\rm circ}$). Fitting a double Gaussian to the observed VPs reveals a hot, mildly rotating component, and a rapidly rotating, cold component. The mean rotation of the latter component is an excellent measure of the mean streaming of the nuclear disc, and therefore of the circular velocity. In turn, this can be used to constrain the central mass density.

We investigated the influence of seeing convolution on the rotation velocity of the disc as measured by fitting a double Gaussian to the VPs. The results are shown in Figure 14 for model 3.1 ($\alpha = -1.0, \Lambda = 1.0$). We plot $\omega_d \equiv v_{\rm disc}/v_{\rm circ}$ as function of the scaling parameter χ for three different radii: $R = 0.5$, 1.0, and 2.0 times the scale length of the nuclear disc. Seeing convolution decreases ω_d considerably, especially at $R \lesssim 2R_d$. For PSFs whose FWHM exceeds $2R_d$ (i.e., $\chi > 2$), the VPs no longer reveal two separate components, and the double Gaussian fit becomes meaningless. The fact that $\omega_d \sim 0.8$ at $R = 0.5R_d$ for $\chi = 0$ (i.e., for the unconvolved VPs), is due to the fact that at small radii the cusped bulge dominates the surface brightness, and therefore these central VPs do not clearly reveal two components, making the interpretation of the double Gaussian fit in terms of a disc- and bulge-component somewhat ambiguous. This is not the case for a non-cusped bulge ($\alpha = 0.0$), where, as shown in Figure 9, the disc dominates the central VPs, and has $\omega_d \approx 0.95$.

7 THE INFLUENCE OF A NUCLEAR BLACK HOLE

Many, if not all, galaxies might harbour massive BHs in their nuclei. In order to study their kinematic signature in the presence of a nuclear disc, we have constructed several models with central BH (see Table 1). A massive BH in the nucleus of a galaxy strongly influences the central dynamics. Early studies based on spherical models showed that adiabatic growth of the BH results in a power-law cusp in the stellar density profile with a logarithmic slope of central surface brightness in the range from $-1/2$ to $-5/4$. Furthermore, hydrostatic equilibrium requires the RMS velocity of the stars surrounding the BH to have a $r^{-1/2}$-cusp (Bahcall & Wolff 1976; Young 1980; Quinlan et al. 1995).

QZMH showed that oblate (α, β)-models with a BH are physical (i.e., have non-negative DF) when $\alpha \leq -0.5$. Under this condition a nuclear disc can be added, while maintaining a positive DF. At sufficiently small radii the potential is dominated by the BH, and an explicit expression for $\rho(\Psi, R)$ can be obtained. Therefore, an asymptotic expression for $f_e(E, L_z)$ can be calculated in terms of elementary functions (see Appendix A).

We have added BHs of both $M_{\rm BH} = 0.1 M_{\rm disc}$ and $M_{\rm BH} = M_{\rm disc}$ to the model with $\alpha = -1.0$, both for the model with and without nuclear disc (see Table 1 for the parameters). The real moments of the VPs after seeing convolution are shown in Figure 15. With this parameterization of the VPs the signature of the BH with mass $0.00303 M_{\rm bulge}$ is limited to a small increase of the velocity dispersion provided that $\chi \lesssim 0.5$. The signature of a BH that is ten times more massive, is much more pronounced. Not only is a strong central increase of velocity dispersion visible (even for $\chi = 1$), but in addition, the mean rotation is considerably larger than in the case without BH. There is no major difference of BH-signature between the cases with and without nuclear disc, as judged from the four velocity profile moments presented here. This is due to the fact that the nuclear disc only adds a small fraction of the light to the VPs, especially at small radii, where the cusp dominates.

We have also fitted double Gaussians to the seeing convolved VPs. Figure 16 shows the parameter ω_d as function of radius in arcseconds for three different values of χ for models 3.1 (solid lines), 3.3 (dashed lines), and 3.5 (dotted lines). The main result is that for the model with $M_{\rm BH} = M_{\rm disc}$ (model 3.5) the rotation velocity of the nuclear disc as measured from the double Gaussian fit remains over 90% of the circular velocity down to $0.5R_d$ as long as $\chi \lesssim 1$. The fact

that one can actually measure this is due to the fact that the circular velocity in the centre is so large that the small part of the VP that is due to the nuclear disc, clearly shows up in the wing of the bulge part. In the case of the less massive BH (model 3.3), the effect of the BH on the circular velocity is so small that the disc part of the VP remains hidden in the dominating bulge part, so that the double Gauss fit is rather ambiguous, and can no longer be interpreted as due to a disc- and bulge-component.

Since the circular velocity at $0.25R_d$ for the case with $M_{\rm BH} = M_{\rm disc}$ is 4.2 times larger than for the case without BH, measuring $v_{\rm circ}$ so close to the centre provides an excellent measure of the central mass-density and will immediately reveal the presence of the BH, without the ambiguity with respect to velocity anisotropies that hampers the interpretation of an observed, central increase of velocity dispersion. Therefore, nuclear discs can be used to put unambiguous constraints on the presence of nuclear BHs in these galaxies.

8 SUMMARY AND DISCUSSION

We have constructed axisymmetric, two-integral models of elliptical galaxies harbouring small, nuclear discs. We used the HQ method to calculate the even part of the phase-space distribution function (DF) for models that consist of an (α, β)-spheroid (the bulge) which contains a highly flattened exponential spheroid (the disc). In addition, we considered the effects of a massive nuclear BH. The bulge has a central power-law density cusp ($\rho \propto r^{\alpha}$). The disc is thick, and has an exponential surface brightness profile. Although we concentrated on nuclear discs, the models presented here can equally well be used to describe, e.g., S0 galaxies, by simply increasing the horizontal scale length of the disc.

For models with only a moderate cusp (i.e., $\alpha > -0.5$), there is a region in the plane of disc-to-bulge ratio Δ versus disc scale-length R_d where the even part of the DF is not strictly positive. For the cusped models investigated here ($\alpha = -0.5, -1.0, -1.5$) no such region in ($\Delta$, R_d) parameter space exists, so all these models are physical. We have considered different odd parts of the DF, which lead to models with a large variety of streaming motions. Upon defining the odd part of disc and bulge separately, we obtain disc- and bulge-components that are dynamically decoupled. As an example of such systems, we have constructed models with counter-rotating nuclear discs.

Our models are locally stable against axisymmetric perturbations according to the Toomre criterion. The nuclear discs are stable as long as they are not too flattened or compact. The presence of a central cusp strongly increases the stability of the nuclear disc. Our two-integral nuclear discs are also stable against bar formation. Steeper cusps of the bulge ensure better stability. The central surface brightness of the disc with disc-to-bulge ratio such that the disc is only just locally stable over its entire extension depends on the value of the disc scale-length R_d. The relation is remarkably similar to the observed μ_0-R_d relation for a large sample of spirals, S0s, and discy ellipticals. If stability arguments are indeed responsible for this relation, this suggests that discs build up their mass until they become marginally stable.

We have investigated the effects of seeing convolution on the kinematic observables of nuclear discs. We considered PSFs with FWHM in the range of $0.5R_d$ up to $5R_d$. If $R_d = 0.2''$, as observed in a number of Virgo ellipticals, this corresponds to the range from $0.1''$ (i.e., HST resolution) up to $1''$ (i.e., typical ground-based resolution). For $\chi \equiv {\rm FWHM}/R_d \gtrsim 3$ the nuclear disc becomes kinematically undetectable. This is similar to the value of χ for which the disc becomes photometrically undetectable. One does, however, observe a slightly larger central velocity dispersion than would be the case without nuclear disc. This is due to seeing convolution of the rotation curve of the nuclear disc. It has been argued by a number of authors that this effect might mimic the presence of a central BH. However, we find that the effect is rather small, not exceeding 10%, and conclude that nuclear discs can *not* mimic the presence of a massive BH.

In the case of counter-rotating nuclear discs the counter-rotation is only unambiguously detectable from the VPs if the seeing FWHM is smaller than $\sim 2R_d$ (for $\Lambda = 1$). For such small seeing FWHM, the central line-of-sight velocity profile is dominated by the disc, and therefore one observes in addition to the counter-rotation a central *decrease* in velocity dispersion. It is surprising that in most cases where counter-rotation is observed, one finds an additional strong, central *increase* in velocity dispersion. Although we note that this might be an indication for a nuclear BH, in which case the high central dispersion is due to seeing convolution of the Keplerian rotation curve, further dynamical modeling is required to confirm this.

When fitting a double Gaussian to the VPs one extracts information of the kinematics of both components. Although the assumption that both components have Gaussian VPs is clearly simplified, it gives a first order estimate of the rotation velocities of the disc. The disc rotates with almost the circular velocity ($v_{\rm disc} \approx 0.95\ v_{\rm circ}$). The relative contribution of disc- and bulge light to the VP depends strongly on radius, and on the parameters of the model. All constructed models have disc-to-bulge ratio chosen so that at $R = R_d$ the disc contributes 50% to the light of the VP (i.e., $\Lambda = 1$). Therefore, the relative disc contribution to the VPs at $R < R_d$ decreases strongly with increasing cusp steepness of the bulge. For shallow cusps, the disc still contributes a significant amount of light to the central VPs and a central decrease in velocity dispersion is observed. For more steeply cusped bulges, the disc becomes only detectable at somewhat larger radii.

The fact that the nuclear disc rotates with almost the circular velocity has important implications. Measuring the mean rotation of the nuclear disc allows an accurate determination of the central density (or central mass-to-light ratio) of its host galaxy. We have tried to "measure" the rotation of the nuclear disc by fitting a double Gaussian to the seeing convolved VPs. We compared these results to the mean rotation $v_{\rm disc}$ as calculated from equation (4.5). For $R \gtrsim 2R_d$ and $\chi \lesssim 1$ we found the mean of the rapidly rotating Gaussian to be in good agreement (within a few percent) with $v_{\rm disc}$. There VPs can therefore be used to put strong constraints on the central mass-to-light ratio of the host galaxy. At smaller radii the cusp dominates and the seeing convolution has important influences. Therefore, these central VPs do not clearly show two distinct components and the double Gaussian fit is somewhat arbitrary. The fit can no longer be interpreted as representing the disc- and bulge component.

We have also included a massive BH in the centre of these models to investigate whether nuclear discs can be used to test for the presence of such BHs in the same way as ionized gas discs can. In both the cases with and without nuclear disc a strong central increase in velocity dispersion is observed (for a BH that is massive enough). The observed rotation velocities, as measured from the first moment of the VP, increases. We have fitted double Gaussians to the VPs, to see if we can measure the rotation velocity of the nuclear disc. Since the disc rotates with almost the circular velocity, such measurements provide an accurate estimate of the central mass density of the galaxy. We find that for a BH whose mass equals that of the nuclear disc, the circular velocity becomes so large close to the centre, that even the VPs at only 0.5 R_d from the centre clearly reveal two components. The mean velocity of the rapid rotating component is an excellent indicator of the circular velocity. Since at these small radii, $v_{\rm circ}$ is considerably larger than for the case without BH, the observed rotation velocity of the nuclear disc provides strong evidence for the presence of the BH. Most importantly, interpretation of these rotation velocities is far less ambiguous than interpretation of a central increase of velocity dispersion.

Although the stellar dynamics of the disc and the bulge is likely to be influenced by a third integral of motion, we suspect that the results discussed here, based upon two-integral models, are not strongly influenced by this oversimplification. It is well known that when two-integral models require the presence of a massive BH to fit a central increase of velocity dispersion, one always has the possibility that a three-integral model without BH might equally well fit the data. However, if one observes a nuclear disc whose rotation velocity exceeds the circular velocity that would correspond to a constant mass-to-light ratio model, invoking a third integral of motion can not alter the conclusion that there has to be a central increase in mass-to-light ratio.

The high spatial-resolution spectra obtainable with HST can be used to accurately measure the rotation curve of the nuclear discs, giving an excellent measure of the central density of the galaxy. Although the spectral resolution of the Faint Object Spectrograph (FOS) aboard the HST may be too poor to resolve the VPs in enough detail to allow for $v_{\rm disc}$ to be measured, observations with the Space Telescope Imaging Spectrograph (STIS), to be installed during the next servicing mission, will allow the VPs to be measured with high enough both spatial, and spectral resolution to put stringent constraints on the presence of possible massive BHs in the nuclei of these galaxies.

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APPENDIX: ASYMPTOTIC APPROXIMATION OF EVEN PART OF DF CLOSE TO BH

At sufficiently small radii the density of the composite disc/bulge system can be approximated by

$$\rho(R,z) = \rho_{0,b}\left(\frac{m_b}{R_b}\right)^{\alpha} - \rho_{0,d}\ln\left(\frac{m_d}{R_d}\right), \tag{A1}$$

where m_b and m_d are the constant density spheroids of bulge and disc respectively. As long as $\alpha > -2$ and $\alpha + 2\beta < -2$, the central potential can be approximated by

$$\Psi(R,z) = \frac{GM_{\rm BH}}{\sqrt{R^2+z^2}} + \Psi_0^*. \tag{A2}$$

Here Ψ_0^* is the finite, central stellar potential given by the sum of $\psi_{0,b}$ (equation 2.9) and $\psi_{0,d}$ (equation 2.18). Solving z^2 from equation (6.7) and substitution in equation (A1) yields an explicit expression of $\rho(\Psi, R)$. Using the alternative form of the contour integral given by equation (3.3) in HQ, and integrating along a circular contour parameterized by angular parameter θ (i.e., $\Psi = E + Ee^{i\theta}$, $-\pi \le \theta \le \pi$), an analytic solution for the approximating $f_e(E, L_z) = f_e^b(E, L_z) + f_e^d(E, L_z)$ is found.

For $\alpha = -1$ and $\alpha + 2\beta = -4$ we find

$$f_e^b(E,L_z) = \frac{\rho_{0,b}q_b}{\sqrt{8}\pi^2\Phi_b^{3/2}}\left(\frac{E-\Psi_0^*}{\Phi_b}\right)^{-1/2}\frac{1+\xi_b}{(1-\xi_b)^2}, \tag{A3}$$

where $\xi_b = (1-q_b^2)\eta^2$, and $\Phi_b = GM_{\rm BH}/R_b$ (see also Dehnen and Gerhard 1994, and QZMH). Similarly, for the disc part

$$\begin{aligned} f_e^d(E,L_z) = {} & \frac{\rho_{0,d}}{2\sqrt{8}\pi^2\Phi_d^{3/2}}\left(\frac{E-\Psi_0^*}{\Phi_d}\right)^{-3/2} \times \\ & \left[g(\xi_d) - \ln\left(\frac{E-\Psi_0^*}{\Phi_d}\right) - \ln q_d - 2\ln 2 - 3\right], \end{aligned} \tag{A4}$$

where

$$\begin{aligned} g(\xi_d) = {} & \frac{3-2\xi_d}{1-\xi_d} + \\ & \left[1 - \frac{3-4\xi_d}{(1-\xi_d)^2}\right]\sqrt{\frac{1-\xi_d}{\xi_d}}\arctan\sqrt{\frac{\xi_d}{1-\xi_d}}. \end{aligned} \tag{A5}$$

Here $\xi_d = (1-q_d^2)\eta^2$, and $\Phi_d = GM_{\rm BH}/R_d$. For the radial, $\eta = 0$ orbits $g(\xi_d) = 1$.